Dissipative sandpile models with universal exponents

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We consider a dissipative variant of the stochastic-Abelian sandpile model on a two-dimensional lattice. The boundaries are *closed* and the dissipation is due to the fact that each toppled grain is removed from the lattice with probability ϵ. It is shown that the scaling properties of this model are in the universality class of the stochastic-Abelian models with conservative dynamics and *open* boundaries. In particular, the dissipation rate ϵ can be adjusted according to a suitable function $\epsilon = f(L)$, such that the avalanche size distribution will coincide with that of the conservative model on a finite lattice of size L.

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I. INTRODUCTION

Sandpile models have been studied extensively as a paradigm of self-organized criticality (SOC) [1–3], a theoretical framework that describes nonequilibrium systems that spontaneously evolve into a critical state. At the critical state these systems exhibit avalanche dynamics with long-range spatial and temporal correlations, which resembles the behavior at equilibrium critical points [4–16]. The relevance of SOC in the context of earthquakes, avalanches in granular flow and mass extinctions was considered [17]. To examine the dependence of the critical state on properties of the models, different sandpile models have been introduced, in addition to the deterministic model of Bak, Tang, and Wiesenfeld (BTW) [1]. These include the Manna model, which is stochastic [18]. In contrast with earlier results [19–22], numerical simulations using an extended set of critical exponents showed that deterministic and stochastic models exhibit different scaling properties and thus belong to different universality classes [23–25]. The crossover between the two classes was also studied [26]. Further support for this result was obtained using moment analysis [27] and multifractal analysis [28]. It was found that the avalanche size distribution of the BTW model exhibits multifractal scaling [28–30]. This indicates that an ordinary finite size scaling analysis is not sufficient to describe the scaling behavior of the BTW model. However, finite size scaling applies in the case of the Manna model.

In the models mentioned above the dynamical rules are conservative, and therefore they are referred to as conservative sandpile models. In these models dissipation takes place only through the boundaries. This dissipation is essential for the self organization of the system into the critical state, which is reached through the balance between the driving rate and the dissipation rate in the system. The connection between this critical state and ordinary critical phenomena was further established by considering sandpile models as closed systems with no dissipation [31–33]. The closed systems do not exhibit self-organized criticality and the critical state is reached by tuning the energy density to its critical value, as in equilibrium critical phenomena. A deterministic sandpile model with bulk dissipation was introduced and studied by Olami, Feder, and Christensen (OFC) [34]. It was found to exhibit nonuniversal scaling exponents which depend on the dissipation rate. More recent studies reexamined the criticality of the OFC model [35–37].

In this paper we consider a dissipative variant of the Manna model. The model is defined on a closed lattice and the dissipation takes place in the bulk. Unlike the OFC model, it exhibits universal exponents which do not depend on the dissipation rate. However, the scaling range is controlled by the dissipation, and broadens as the dissipation decreases. This result provides insight on the role of dissipation. In open systems the boundaries tune the dissipation such that the correlation length coincides with the system size. As a result, criticality is observed over the entire system. Using closed systems with bulk dissipation we decouple the dissipation rate from the system size and uncover a crucial tuning mechanism in self-organized criticality.

The paper is organized as follows. Conservative sandpile models with open boundaries are reviewed in Sec. II. In Sec. III we introduce bulk dissipation into these models and examine its effects. The scaling properties of the conservative and dissipative models are analyzed in Sec. IV. The results are discussed in Sec. V and summarized in Sec. VI.

II. CONSERVATIVE SANDPILE MODELS

Consider a sandpile model on a two dimensional square lattice of linear size L. Each site $\mathbf{i}$ is assigned a dynamic variable $E(\mathbf{i})$ which represents some physical quantity such as energy, grain density, or strain. A configuration $\{E(\mathbf{i})\}$ is *stable* if for all sites $E(\mathbf{i}) < E_c$, where E_c is a threshold value. Starting from a stable configuration, the system is perturbed by adding an amount of energy δE to a randomly chosen site $\mathbf{i}$. When $E(\mathbf{i})$ becomes larger than the threshold E_c, the configuration becomes unstable. The site $\mathbf{i}$ topples and its energy is distributed according to

$$E(\mathbf{i}) \rightarrow E(\mathbf{i}) - \sum_{\mathbf{e}} \Delta E(\mathbf{e}),$$

$$E(\mathbf{i}+\mathbf{e}) \rightarrow E(\mathbf{i}+\mathbf{e}) + \Delta E(\mathbf{e}), \qquad (1)$$

where $\mathbf{e}$ are a set of vectors from the site $\mathbf{i}$ to its nearest neighbors. The redistributed energies $\Delta E(\mathbf{e})$ are elements of a vector ΔE, to be termed the relaxation vector. For a square lattice with relaxation to nearest neighbors it is of the form

$\Delta E=(E_N,E_E,E_S,E_W)$, where E_N,E_E,E_S, and E_W are the amounts transferred to the northern, eastern, southern, and western nearest neighbors, respectively. As a result of the relaxation, $E(\mathbf{i}+\mathbf{e})$ for one or more of the neighbors may reach or exceed the threshold E_c. The relaxation rule is then applied until a stable configuration is obtained. The resulting sequence of topples is an avalanche which propagates through the lattice. The boundaries of the system are open. Therefore, when an edge site becomes unstable, energy is dissipated through the boundary.

An important classification of sandpile models is between Abelian and non-Abelian models [5,38]. Consider a stable initial configuration and a series of two avalanches initiated by adding energy δE to sites **1** and **2**. The model is Abelian if the resulting stable configuration after the two avalanches is independent of the order in which the energy was added to **1** and **2** as well as of the order in which unstable sites are toppled. Any dependence on the order makes the model non-Abelian.

An avalanche can be characterized by its size, s, which is the total number of toppling events that occurred during the course of the avalanche, and by its area, a, which is the total number of sites that experienced at least one toppling event. In the critical state the avalanche size distribution $P(s)$ was found to exhibit the power-law form

$$P(s) \sim s^{-\tau_s}, \tag{2}$$

within a scaling range which depends on the system size L, where τ_s is a critical exponent. It was found that the critical exponents depend on the relaxation vector ΔE. In the BTW model the toppling rule is deterministic, with $E_c=4$, $\delta E=1$ and $\Delta E=(1,1,1,1)$. When an active site with $E(\mathbf{i})>4$ is toppled, it still distributes only four grains to its nearest neighbors and thus does not become empty after the topple had occurred. The BTW model was shown to be Abelian [5].

In a class of stochastic sandpile models, first introduced by Manna, a set of neighbors is randomly chosen for relaxation [18]. Such models can be specified by a set of relaxation vectors, each vector being assigned a probability for its application. In the original model introduced by Manna [18], an unstable site distributes all its energy to its neighbors and becomes empty. It can be easily shown that this model is non-Abelian. Consider two nearest neighbor sites **1** and **2** which are unstable simultaneously, namely $E(\mathbf{1})\geq E_c$ and $E(\mathbf{2})\geq E_c$. If we first topple site **1** and then topple site **2** (assuming no further sites becomes unstable), in the resulting configuration $E(\mathbf{2})=0$ while $E(\mathbf{1})\geq 0$. If the order of topplings is reversed one obtains $E(\mathbf{1})=0$ and $E(\mathbf{2})\geq 0$, namely, the resulting configuration may depend on the order of topplings and thus the model is non-Abelian.

Later, Abelian variants of the Manna model were studied, in which the number of energy units distributed from an unstable site is a constant [23,33,39]. To see that these models are Abelian consider two simultaneously unstable nearest neighbor sites **1** and **2**. Assume that the relaxation vectors of these sites are predetermined (randomly) and are independent of the order of their toppling. In this case, just as in the BTW model, the resulting configuration is independent of the order of topplings [39]. Note however, that unlike the BTW model in which the final configuration is fully determined by the initial unstable configuration, in the Manna model the final configuration depends also on the random choice of the relaxation vectors. Another property of the Manna model is that the relaxation vector is isotropic only on average, while the single topplings are anisotropic.

Here we focus on the Manna two-state model which is a stochastic model with $E_c=2$ and $\delta E=1$. In this model, when an unstable site is toppled, two grains move to randomly chosen nearest neighbor sites, with no correlations between the destinations of the two grains. This amounts to six relaxation vectors of the form $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1),\ldots$, each of them drawn with probability of $1/8$, and four relaxation vectors of the form $(2,0,0,0)$, $(0,2,0,0),\ldots$, each drawn with probability of $1/16$.

The average avalanche size $\langle s\rangle_L$, can be found from an analogy between sandpile models and random walkers on finite lattices [40]. In the sandpile model, each avalanche starts with the addition of a single grain at a random site. In the critical state, on average, the number of grains leaving the system (through the open boundaries) is equal to the number of grains added to the system. Therefore, on average, the system loses one grain per avalanche through the boundaries.

Consider a grain that is added to the system. It is deposited at a random site on the lattice and may initiate an avalanche. The grain stays in the system during many avalanches. Most of the time it does not move. It moves only when it shares its site with another grain, making the site unstable. In this case it hops to a randomly chosen nearest neighbor. Although the hopping of the grain is intermittent, its path is equivalent to that of a random walker. The path terminates upon the first time that the grain encounters the boundary of the system. Thus, the average number of hops such grain performs is equal to the average path length $\langle n\rangle_L$ of a random walker starting from a random site on the lattice to the boundary. Since each avalanche starts by the addition of a single grain, the average number of hops per avalanche is also equal to $\langle n\rangle_L$. In the Manna two-state model, each toppling event consists of two hopping moves. Thus, the average avalanche size is $\langle s\rangle_L=\langle n\rangle_L/2$. From numerical simulations of random walkers it was found that [40]

$$\langle s\rangle_L = (c_1 L + c_2 L^2), \tag{3}$$

where $c_1=0.28$ and $c_2=0.07$, in agreement with earlier results [5]. Note that this connection between the sandpile models and random walkers would not apply for non-Abelian models, because in these models the number of grains distributed per avalanche is not constant. However, a result of the form of Eq. (3), up to a rescaling of the parameters, applies in the case of the BTW model, in spite of the fact that it is a deterministic rather than a stochastic model [5,40].

III. THE DISSIPATIVE MANNA MODEL

In this paper we consider a dissipative variant of the Manna two-state model. Unlike the conservative Manna model described above, it exhibits dissipation in the bulk, while the boundaries are closed or periodic. More specifically, each time a grain is toppled it has a probability ϵ to be removed from the system. The system evolves into a steady state in which the energy added to the system per avalanche, is balanced, on average, by the energy that flows out by the dissipation. Therefore, the average amount of energy leaving the system per avalanche is $\delta E = 1$. In the critical state the avalanche size distribution $P(s)$ for this model was found to exhibit the power-law form given by Eq. (2) within a scaling range which is determined by the value of ϵ.

The path of each grain on the lattice, starting from the site in which it was deposited and ending in the site through which it is removed by dissipation is a random walk. Each time a grain hops it has a probability ϵ to be removed. Therefore, the average number of hops a grain performs before it is removed is $1/\epsilon$. In the critical state, on average, one grain is added and one grain is removed per avalanche. Therefore, the average number of hops of grains per avalanche is $1/\epsilon$. Since each toppling event consists of two hops, the average avalanche size is given by $\langle s \rangle_\epsilon = (2\epsilon)^{-1}$. Comparing this result to Eq. (3) we obtain a quantitative connection between the conservative and the dissipative models. The connection is that a dissipative model with dissipation rate

$$\epsilon = \frac{1}{2(c_1 L + c_2 L^2)} \tag{4}$$

exhibits the same average avalanche size as a conservative model on an open lattice of size L. Below we present a detailed comparison between the critical properties of the conservative and dissipative Manna models.

IV. SIMULATIONS AND RESULTS

Setting ϵ in the dissipative model according to Eq. (4) adjusts its average avalanche size to coincide with that of the conservative model on a lattice of size L with open boundaries. One may ask whether such adjustment would also give rise to identical scaling properties of the two models. To examine this question we have performed numerical simulations of both models. Note that in order avoid spurious finite size effects, the simulations of the dissipative model were performed on a lattice of size much larger than L.

In Fig. 1(a) we present the avalanche size distribution $P(s)$ for systems of sizes $L=64$, 128, 256, and 512, for the conservative Manna model with open boundaries (symbols). We also show the avalanche size distribution for the dissipative model with values of ϵ given by Eq. (4) for these values of L (solid lines). Excellent agreement is obtained between the two models for all system sizes both for the critical exponents, within the scaling range, and for the shapes of the curves in the tails beyond the scaling range. Note that in Fig. 1(b) we present the avalanche area distributions $P(a)$ for the conservative Manna model and for the dissipative model, for the same set of values of L and ϵ. The graphs for the two models coincide almost perfectly in the scaling range, with some deviations in the tails. It seems that these deviations are due to the fact that in the conservative model, the avalanche area is strictly limited by L^2, while in the dissipative model there is no strict limit on the avalanche area. Note that for the avalanche sizes, shown in Fig. 1(a), there is no strict limit in both models, which may explain the excellent agreement even in the tails.

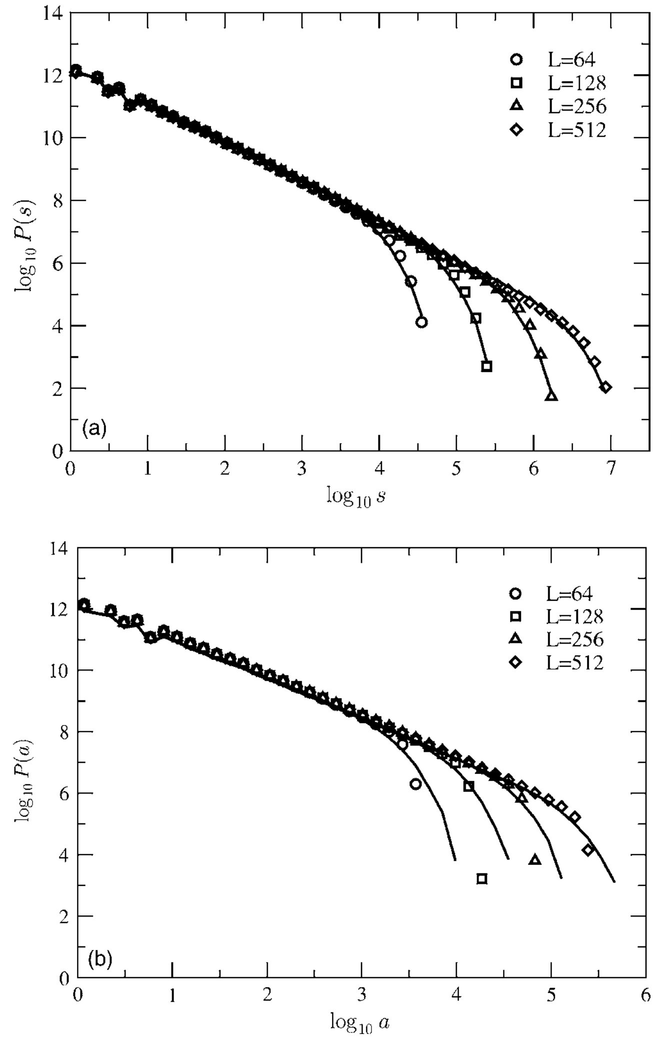

FIG. 1. (a) The avalanche size distributions $P(s)$; and (b) the avalanche area distribution for the (conservative) two-state Manna model with open boundaries (symbols) with $L=64$, 128, 256, and 512 and for the dissipative model (solid lines) with the corresponding values of ϵ, namely, 1.64×10^{-3}, 4.23×10^{-4}, 1.07×10^{-4}, 2.70×10^{-5}, respectively, according to Eq. (4).

The distributions of the avalanche sizes and areas can be related to each other by the conditional expectation value $E[s|a]$ of the avalanche size s given that its area is a. This expectation value turns out to satisfy

$$E[s|a] \sim a^{\gamma_{sa}} \tag{5}$$

with an exponent γ_{sa}. In Fig. 2 we present $E[s|a]$ vs a, for the conservative model (symbols) and for the dissipative model (solid line). Very good agreement is observed within the scaling range. The deviations near the upper cutoff correspond to the deviations found near the cutoffs in Fig. 1(b).

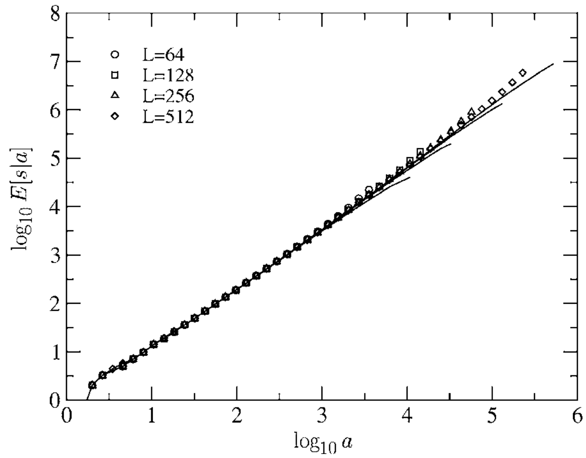

FIG. 2. The conditional expectation value $E[s|a]$ for an avalanche to include s toppling evens given that its area is a for the conservative (symbols) and the dissipative (solid lines) models, with the same values of L and ϵ as in Fig. 1.

Thus, we conclude that these two models exhibit the same critical exponents.

Although their critical exponents are the same, one can identify some differences between the conservative and dissipative models. In particular, conservative models with open boundaries exhibit deviations from homogeneity across the lattice. In Fig. 3 we present the average density ρ of grains in lattice sites vs their distance from the boundary, for the stable configurations obtained after the completion of an avalanche. It is found that the density is strongly depleted for sites adjacent to the edges. Such inhomogeneities tend to slow down the convergence of finite size scaling analysis, and cause uncertainties about the values of critical exponents. In attempt to control the boundary effects, two classes of avalanches were identified: dissipative avalanches that reach the boundaries and dissipate energy and conservative avalanches that do not reach the boundaries [9]. Analysis of the conservative avalanches alone provided useful insight, however, at the cost of reducing the amount of data available, particularly for large avalanches. The dissipative model considered here is translationally invariant and the density of grains is homogeneous throughout the lattice. This may provide better convergence in a variety of finite size scaling calculations.

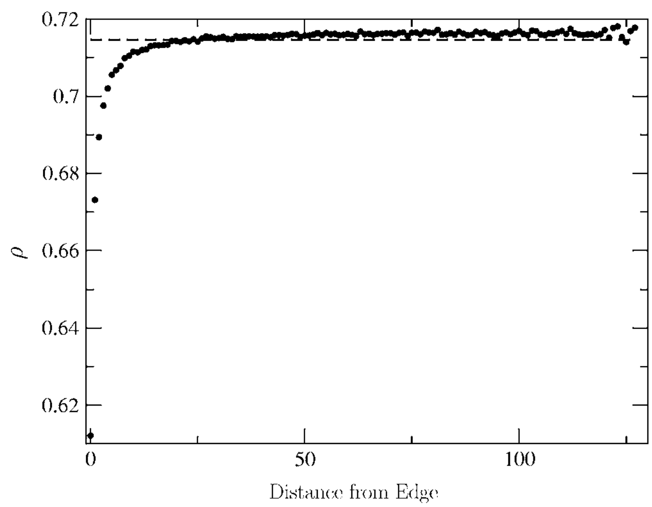

FIG. 3. The average density of grains, ρ, in lattice sites, for stable configurations after the completion of avalanches, as a function of the smallest distance from the boundary in the conservative Manna model with open boundaries (circles). Sites that are adjacent to the boundary are depleted from grains, causing inhomogeneities in the system. The dashed line shows the average density for the dissipative Manna model.

V. DISCUSSION

Studies of sandpile models with closed boundaries, no drive and no dissipation revealed that they exhibit a transition at the critical density ρ_c [31–33]. Above ρ_c, there is an active phase with an infinite sequence of toppling events. Below ρ_c it approaches an absorbing phase in which the configuration is frozen. The transition can be characterized by an order parameter, which is the average number of unstable sites. The critical exponents of the transition in the closed system were calculated numerically [33] and related to those of the sandpile model with open boundaries [41].

In sandpile models with open boundaries the critical state with density ρ_c is reached through the balance between the driving and the dissipation, in the limit of slow driving [33]. For $\rho > \rho_c$ large avalanches appear and the dissipation becomes dominant. As a result, the system is pushed toward ρ_c from above. For $\rho < \rho_c$ avalanches are small and the driving is dominant. The system is thus pushed toward ρ_c from below. This way, the self-organized critical state with grain density ρ_c is stabilized.

The dissipation, which is essential for reaching the critical state, is usually considered as a boundary effect. In this paper we have shown that the dissipation can be incorporated into the model itself without using the boundaries. In the model considered here, the dissipation is stochastic and takes place only in one out of a large number of toppling events. This enables a large number of small avalanches to be effectively conservative, much as those avalanches that do not touch the boundary in the open system. As a result, the criticality is maintained and the exponents remain universal. The dissipation plays a role only in setting the cutoffs of the scaling range, but does not affect the exponents.

These results are in contrast with the OFC model in which the critical exponents were found to be nonuniversal. In the OFC model the dynamical variable is continuous. In the beginning of each avalanche, the same amount of energy is fed to all sites. The dissipation is deterministic. Moreover, in each toppling a constant amount of energy is lost. As a result, the dissipation plays a prominent role at all length scales of the system, from the smallest to the largest avalanches. The exponents are nonuniversal and their values depend on the dissipation rate. In other models, it was found that nonuniversal behavior can be associated with nonlocality [42], long-range moves [43], or certain forms of stochasticity [44].

A similar analysis is expected to apply in the case of deterministic sandpile models, such as the BTW model. However, the avalanche size distribution in the BTW model exhibits multifractal scaling rather than simple finite-size scaling [28], and thus cannot be fully characterized by a single exponent.

VI. SUMMARY

We have analyzed dissipative sandpile models with closed boundaries which give rise to the same universal exponents as the commonly studied conservative models with open boundaries. These models demonstrate that the dissipation can be incorporated into sandpile models rather than introduced as a boundary effect. The dissipation rate serves as a control parameter, which determines the scaling range, in analogy to the role of the system size in the conservative sandpile models.

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